

ON SOME GRAPHS ASSOCIATED WITH THE FINITE ALTERNATING GROUPS

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ABSTRACT. Let $P_0(A_n)$, $\tilde{P}_0(A_n)$, $P_0(\mathcal{T}(A_n))$ and $\mathcal{O}_0(A_n)$ be respectively the proper power graph, the proper quotient power graph, the proper power type graph and the proper order graph of the alternating group A_n , for $n \geq 3$. We determine the number of the components of those graphs. In particular, we prove that the power graph $P(A_n)$ is 2-connected if and only if the power type graph $P(\mathcal{T}(A_n))$ is 2-connected, if and only if either $n = 3$ or none of $n, n-1, n-2, \frac{n}{2}$ and $\frac{n-1}{2}$ is a prime. We also give some information on the properties of those components.

1. INTRODUCTION

In the recent literature in group theory there are many examples of the usefulness of associating a graph $X(G)$ with every finite group G , requiring that isomorphic groups determine isomorphic corresponding graphs. Classic choices for $X(G)$ are: the prime graph, the prime graph for conjugacy class sizes or for complex irreducible characters degrees, the commuting graph, the non-commuting graph, the power graph and, of course, Cayley graphs. The process of passing from G to $X(G)$ reduces the complexity, by focussing on some aspects of G and allows the use of methods from graph theory. If the reduction of complexity is weak, little information about G is lost and one can recover the group G , up to isomorphism, from the graph $X(G)$. This very desirable situation of course depends on the classes of groups under consideration. To be more precise, let \mathcal{F} be the class of the finite groups and let \mathcal{U} and \mathcal{V} be subclasses of \mathcal{F} such that $\mathcal{U} \subseteq \mathcal{V} \subseteq \mathcal{F}$. Assume that, for every $G \in \mathcal{V}$, a graph $X(G)$ is given. Define the class of groups in \mathcal{U} that are *X-recognisable* in \mathcal{V} by

$$\mathcal{R}_X(\mathcal{U}, \mathcal{V}) = \{G \in \mathcal{U} : \forall H \in \mathcal{V}, X(H) \cong X(G) \Rightarrow H \cong G\}.$$

We can measure the level of adherence of X to \mathcal{U} , with respect to \mathcal{V} , by the broadness of $\mathcal{R}_X(\mathcal{U}, \mathcal{V})$ with respect to \mathcal{U} . The best level of adherence is realized for those choices of X such that $\mathcal{R}_X(\mathcal{U}, \mathcal{V}) = \mathcal{U}$; in that case we say that X recognises \mathcal{U} in \mathcal{V} . Frequent applications of that concept in the literature are given when $\mathcal{V} = \mathcal{U}$ or $\mathcal{V} = \mathcal{F}$. If $\mathcal{R}_X(\mathcal{U}, \mathcal{U}) = \mathcal{U}$ we say that X recognizes \mathcal{U} ; if $\mathcal{R}_X(\mathcal{U}, \mathcal{F}) = \mathcal{U}$ we say that X recognizes \mathcal{U} among the finite groups. Even though the adherence level is not the best possible, the definition of X is considered to be interesting if $\mathcal{R}_X(\mathcal{U}, \mathcal{V})$ contains known classes of groups, such as nilpotent groups, solvable groups, abelian groups, simple groups and so on.

In this paper, all the considered classes of groups are included in \mathcal{F} and G always denotes a finite group; every graph $X = (V_X, E_X)$ is finite, undirected, simple and reflexive. Having a loop on every vertex simplify many arguments in the general theory developed in [3] to recover the number and the structure of the components of a graph by the knowledge of the components of its quotient graphs.

We are mainly focused on the *power graph* $P(G)$, that is, the graph with vertex set G and having an edge between two vertices if one of them is the power of the other. Many interesting results on power graphs are collected in the survey [2], where a wide list of open problems and references is given. Nowadays, the study of the power graphs is far to be completed and the recent literature shows a growing interest about the graph theoretical properties of $P(G)$. For instance, in [11] it is shown that the power graph has a transitive orientation and a closed formula for the metric dimension of

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$P(G)$ is established. In [10], relying on a fundamental result about groups with isomorphic power graphs ([6, Proposition 1]), the full automorphism group of $P(G)$ is described. In many cases such as [8, 15, 14, 9, 12], those properties are investigated in relation to the group theoretical properties of G . In particular, the groups G for which $P(G)$ is planar or toroidal or projective are classified in [9]; in [12] it is presented a characterization of the chromatic number of $P(G)$ and the groups whose power graphs are uniquely colorable, split or unicyclic are classified. In our opinion, a motivation for the studying of the power graph is given by its realizing a high level of adherence to the class of finite groups. Indeed, by [14, Theorem 1] and [13, Theorem 15], we have:

Theorem 1. *Let \mathcal{U} be the union of the following classes of groups:*

- (1) *the class \mathcal{S} of simple groups ;*
- (2) *the symmetric groups;*
- (3) *the automorphism groups of a sporadic simple group;*
- (4) *the class \mathcal{C} of cyclic groups;*
- (5) *the dihedral groups;*
- (6) *the generalized quaternion groups.*

Then $\mathcal{R}_P(\mathcal{F}, \mathcal{F}) \supseteq \mathcal{U}$.

Theorem 1 says that the power graph recognizes \mathcal{U} among the finite groups. In particular, as a very remarkable fact, the power graph recognizes \mathcal{S} among the finite groups. On the other hand, $\mathcal{R}_P(\mathcal{F}, \mathcal{F}) \neq \mathcal{F}$. Indeed it is easily seen that $\mathcal{R}_P(\mathcal{F}, \mathcal{F})$ has empty intersection with the class of p -groups G , with $|G| \geq p^3$ and $\text{Exp}(G) = p$. First of all note that any p -group G , with $\text{Exp}(G) = p$ admits a partition given by its distinct subgroups. Thus $P(G)$ is formed by $\frac{|G|-1}{p-1}$ complete graphs on $p-1$ vertices and by the edges $\{x, 1\}$, for $x \in G$. In particular, the structure of $P(G)$ depends only on the size of G . Now assume that a certain p -group G of exponent p , with $|G| \geq p^3$, is recognizable among the finite groups. Pick a p -group H of exponent p with $|G| = |H|$, chosen such that only one among G and H is abelian. Then $P(G) \cong P(H)$ while $G \not\cong H$, a contradiction.

The exact description of $\mathcal{R}_P(\mathcal{F}, \mathcal{F})$ is, at the best of our knowledge, still wide open. In particular, does $\mathcal{R}_P(\mathcal{F}, \mathcal{F})$ contains the full class of the almost simple groups?

The previous example also shows that if \mathcal{A} is the class of the abelian groups, then $\mathcal{R}_P(\mathcal{A}, \mathcal{F}) \subsetneq \mathcal{A}$. Note instead that, by [5, Theorem 1], the power graph recognizes the abelian groups, that is $\mathcal{R}_P(\mathcal{A}, \mathcal{A}) = \mathcal{A}$. It follows that $\mathcal{R}_P(\mathcal{A}, \mathcal{F})$ contains the p -groups of order p^2 and so, using Theorem 1, $\mathcal{C} \subsetneq \mathcal{R}_P(\mathcal{A}, \mathcal{F}) \subsetneq \mathcal{A}$. The exact determination of the class $\mathcal{R}_P(\mathcal{A}, \mathcal{F})$ is a further open problem.

The fact that the power graph recognizes \mathcal{S} among the finite groups, indicates that it is possible to shed light on simple groups through the power graph. We know that if $G \in \mathcal{S}$ and $H \in \mathcal{F}$ are such that $P(H) \cong P(G)$, then $H \cong G$. To reach an effective control of being $P(H) \cong P(G)$, when $G \in \mathcal{S}$ is given, we need a deep knowledge of the graph theoretical properties of $P(G)$. Among those, connectivity is one of the most natural. Since $P(G)$ is always connected, the focus is on 2-connectivity. Recall that a graph X is called 2-connected if, for every $x \in V_X$, the x -deleted subgraph $X - x$ is connected. It is immediately checked that $P(G)$ is 2-connected if and only if the *proper power graph* $P_0(G)$, defined as the 1-deleted subgraph of $P(G)$, is connected. The study of 2-connectivity of $P(G)$ for $G \in \mathcal{S}$, begun with a question posed by Pourgholi, Yousefi-Azari and Ashrafi [15, Question 13] about the existence of finite simple groups with 2-connected power graphs. In [1] some simple groups are analyzed and for all of them it is shown that the power graph is not 2-connected. We believe that it could be interesting to find the number $c_0(G)$ of components of $P_0(G)$ for $G \in \mathcal{S}$, the aim being a grouping of simple groups having the same $c_0(G)$. This research echoes the counting of the components of the prime graph for simple groups in [16], and could lead to an analogous new understanding of the simple groups.

In this paper, we compute $c_0(A_n)$ for all $n \geq 3$, and we found among other things, an infinite family of alternating groups with 2-connected power graph. We also add a description of some graph theoretical properties of the components.

2. NOTATION AND MAIN RESULTS

In [4] we studied the 2-connectivity of $P(G)$, for $G \in \mathcal{F}$, as an application of the general theory developed in [3]. Theorem B in [3] gives an explicit formula for counting the components of a graph X by the knowledge of the components of its quotient graph Y when the projection is an orbit homomorphism. We briefly recall the main concepts and definitions needed for the sequel. For further details, the reader is referred to [3].

2.1. Graphs and homomorphisms. For a finite set A and $k \in \mathbb{N}$, let $\binom{A}{k}$ be the set of the subsets of A of size k . In this paper, as in [3] and [4], a graph $X = (V_X, E_X)$ is a pair of finite sets such that $V_X \neq \emptyset$ is the set of vertices and E_X is the set of edges given by the union of the set of loops $L_X = \binom{V_X}{1}$ and of a set $E_X^* \subseteq \binom{V_X}{2}$. Note that while there is a loop on every vertex, the set E_X^* may be empty. If $e \in E_X^*$ we say that e is a *proper edge*. We usually specify the edges of a graph giving only the edges in E_X^* . X is called *complete* if $E_X = \binom{V_X}{1} \cup \binom{V_X}{2}$.

From now on, let $X = (V_X, E_X)$ and $Y = (V_Y, E_Y)$ be two graphs and let $\varphi : V_X \rightarrow V_Y$ be a map. For every $y \in V_Y$, the subset $\varphi^{-1}(y)$ of V_X is called the *fibre* of φ on y . The relation \sim_φ on V_X defined, for every $x, y \in V_X$, by $x \sim_\varphi y$ if $\varphi(x) = \varphi(y)$, is an equivalence relation. The equivalence classes of \sim_φ are called φ -*cells* and coincide with the nonempty fibres of φ . We call \sim_φ the equivalence relation induced by φ and denote the corresponding quotient graph by X/\sim_φ . φ is called a *homomorphism* from X to Y if, for every $x_1, x_2 \in V_X$, $\{x_1, x_2\} \in E_X$ implies $\{\varphi(x_1), \varphi(x_2)\} \in E_Y$. In that case we use the notation $\varphi : X \rightarrow Y$. The set of homomorphisms from X to Y is denoted by $\text{Hom}(X, Y)$. Let $\varphi \in \text{Hom}(X, Y)$. Then φ naturally induces a map from E_X to E_Y , denoted again by φ , associating to every $e = \{x_1, x_2\} \in E_X$ the edge $\varphi(e) = \{\varphi(x_1), \varphi(x_2)\} \in E_Y$. φ is called a *2-homomorphism* if, for every $e \in E_X^*$, $\varphi(e) \in E_Y^*$; *surjective* (injective) if $\varphi : V_X \rightarrow V_Y$ is surjective (injective). If \hat{X} is a subgraph of X , we define the image of \hat{X} by φ as the subgraph of Y given by $\varphi(\hat{X}) = (\varphi(V_{\hat{X}}), \varphi(E_{\hat{X}}))$. φ is called *complete* if $\varphi(X) = Y$. Note that being complete is a stronger than being surjective. We call φ *locally strong* if for every $x_1, x_2 \in V_X$, $\{\varphi(x_1), \varphi(x_2)\} \in E_Y$ implies the existence of $\tilde{x}_2 \in \varphi^{-1}(\varphi(x_2))$ such that $\{x_1, \tilde{x}_2\} \in E_X$; *pseudo-covering* if it is surjective and locally strong; an *orbit homomorphism* if there exists $\mathfrak{G} \leq \text{Aut}(X)$ such that the partition of V_X into φ -cells coincides with the partition of V_X into \mathfrak{G} -orbits. If φ is an orbit homomorphism with respect to \mathfrak{G} we shortly say that φ is \mathfrak{G} -*consistent* or that \mathfrak{G} is φ -*consistent*. Finally φ is called *tame* if every fibre of φ is connected.

2.2. Components, admissibility, vertex deleted subgraphs. A component of X is a maximal connected subgraph of X . The component of X containing the vertex $x \in V_X$ is denoted by $C_X(x)$ or more simply, when no confusion is possible, by $C(x)$. The set of components of X is denoted by $\mathcal{C}(X)$.

Given $U \subseteq V_X$ and $y \in V_Y$, define the *multiplicity* of y in U , through the map φ , by the non-negative integer $k_U(y) = |U \cap \varphi^{-1}(y)|$. We say that y is *admissible* for U (or U is admissible for y), if $k_U(y) > 0$. The set $\varphi(U)$ is therefore the set of vertices of Y admissible for U . If \hat{X} is a subgraph of X we adopt the same language referring to its set of vertices $V_{\hat{X}}$. In particular $k_{\hat{X}}(y)$ is defined by $k_{V_{\hat{X}}}(y)$. Note that $k_X(y)$ is simply the size of the fibre $\varphi^{-1}(y)$. The subgraphs on which we are focused are the components of X . The set of components of X admissible for y is denoted by $\mathcal{C}(X)_y = \{C \in \mathcal{C}(X) : k_C(y) > 0\}$.

A homomorphism $\varphi \in \text{Hom}(X, Y)$ is called *component equitable* if for every $y \in V_Y$ and every $C, \overline{C} \in \mathcal{C}(X)_y$, $k_C(y) = k_{\overline{C}}(y)$. In [3, Propositions 5.9 and 6.9] it is shown that every orbit complete homomorphism is component equitable and pseudo-covering.

Let $x_0 \in V_X$ be fixed. The vertex deleted subgraph is the graph $X_0 = ((V_X)_0, (E_X)_0)$ having vertex set $(V_X)_0 = V_X \setminus \{x_0\}$ and edge set $(E_X)_0$ given by the edges in E_X not incident to x_0 . We use that uniform notation disregarding, from the notational point of view, the particular x_0 used. Moreover we write $\mathcal{C}_0(X)$ for the set of the components of X_0 as well as $c_0(X)$ for their number. In order to improve readability, we occasionally introduce some slight variation of that notational rule.

2.3. The proper power graph and its quotients. The *power graph* of G is the graph $P(G)$ with $V_{P(G)} = G$ and edge set E where, for $x, y \in G$, $\{x, y\} \in E$ if there exists $m \in \mathbb{N}$ such that $x = y^m$ or $y = x^m$. Let $G_0 = G \setminus \{1\}$. The *proper power graph* $P_0(G) = (G_0, E_0)$ is defined as the 1-deleted subgraph of $P(G)$. The number of components of $P_0(G)$ is denoted by $c_0(G)$.

In [4] we got a formula for $c_0(G)$ through the consideration of a series of quotients of $X = P_0(G)$. For a particular class of permutation groups, the so-called fusion controlled permutation groups ([4, Definition 6.1]), that formula became particularly concrete ([4, Theorem A]) by considering a relevant quotient graph, the power type graph. Throughout the paper let n indicate a natural number. The symmetric group S_n and the alternating group A_n are interpreted as naturally acting on the set $N = \{1, \dots, n\}$ and with identity element id . They are both fusion controlled and so they share a similar method for the interpretation and counting of their proper power graph components. Recall that $G \leq S_n$ is called fusion controlled if for every $\psi \in G$ and $x \in S_n$ such that $\psi^x \in G$, there exists $y \in N_{S_n}(G)$ such that $\psi^x = \psi^y$. We present an overview of the quotient graphs, introduced in [4], that we are going to use.

The *quotient power graph* $\tilde{P}(G)$ is the quotient graph of $P(G)$ with respect to the equivalence relation identifying $x, y \in G$ if $\langle x \rangle = \langle y \rangle$. The vertex set of $\tilde{P}(G)$, denoted by $[G]$, is formed by the corresponding equivalence classes $[x]$, for $x \in G$, and the edge set is denoted by $[E]$. We define the order of $[x]$ as the order of x . For every $X \subseteq G$, we also set $[X] = \{[x] \in [G] : x \in X\}$. The *proper quotient power graph* $\tilde{P}_0(G) = ([G]_0, [E]_0)$ is defined as the $[1]$ -cut subgraph of $\tilde{P}(G)$. Note that we have set $[G]_0 = [G] \setminus \{[1]\}$. We recall a very useful link between edges in the proper power graph and in the proper quotient power graph ([4, Lemma 3.5]).

$$(2.1) \quad \forall x, y \in G_0, \{[x], [y]\} \in [E]_0 \text{ if and only if } \{x, y\} \in E_0.$$

The projection $\pi : G_0 \rightarrow [G]_0$, defined by $\pi(x) = [x]$ for all $x \in G_0$ gives a pseudo-covering homomorphism between the graph $P_0(G)$ and its tame quotient $\tilde{P}_0(G)$ ([4, Definition 3.2 and Lemma 3.6]). As a consequence, the number $\tilde{c}_0(G)$ of the components of $\tilde{P}_0(G)$ is equal to $c_0(G)$.

The *order graph* $\mathcal{O}(G)$ is the graph with vertex set $O(G) = \{o(g) : g \in G\}$, where for $a, b \in O(G)$, $\{a, b\} \in E_{\mathcal{O}(G)}$ if a divides b or b divides a . The *proper order graph* $\mathcal{O}_0(G)$ is the $[1]$ -deleted subgraph of $\mathcal{O}(G)$. In [4, Corollary 4.3], we proved that $\mathcal{O}_0(G)$ is a quotient of $\tilde{P}_0(G)$ which usually simplifies in a too drastic way the complexity of $\tilde{P}_0(G)$. In fact, the natural map $\tilde{o} : [G]_0 \rightarrow \mathcal{O}_0(G)$ defined by $\tilde{o}([x]) = o(x)$ for all $[x] \in [G]_0$, induces a complete 2-homomorphism \tilde{o} from $\tilde{P}_0(G)$ onto $\mathcal{O}_0(G)$ ([4, Proposition 4.2]) which is not, in general, pseudo-covering ([4, Example 4.4]). We denote by $c_0(\mathcal{O}(G))$ the number of components of $\mathcal{O}_0(G)$. Note that $\{a, b\} \in E_{\mathcal{O}_0(G)}^*$ if and only if a divides b or b divides a , $a \neq b$ and both a and b are different from 1.

Finally we recall the definition of the power type graph $P(\mathcal{T}(G))$, for $G \leq S_n$ ([4, Section 5.4]). We start recalling the definition of power in the set $\mathcal{T}(n)$ of the partitions of n . Recall that a partition of n is an unordered r -tuple $T = [x_1, \dots, x_r]$, with $x_i \in \mathbb{N}$ for all $i \in \{1, \dots, r\}$ and $r \in \mathbb{N}$, such that $n = \sum_{i=1}^r x_i$. The x_i are the terms of T . Given $T \in \mathcal{T}(n)$, let $m_1 < \dots < m_k$ be its k distinct terms, for some $k \in \mathbb{N}$; if m_j appears $t_j \geq 1$ times in T we use the notation $T = [m_1^{t_1}, \dots, m_k^{t_k}]$ and say that t_j is the multiplicity of m_j . We will accept, in some occasions, the multiplicity $t_j = 0$ simply to say that a certain natural number m_j does not appear as a term in T . We usually omit to write down the multiplicities equal to 1. The partition $[1^n]$ is called the trivial partition and we put $\mathcal{T}_0(n) = \mathcal{T}(n) \setminus \{[1^n]\}$. We define $\text{lcm}(T) = \text{lcm}\{m_i\}_{i=1}^k$ and $\text{gcd}(T) = \text{gcd}\{m_i\}_{i=1}^k$; the order of T is defined by $\text{lcm}(T)$ and written as $o(T)$. For $T = [x_1, \dots, x_r] \in \mathcal{T}(n)$ and $a \in \mathbb{N}$, the power of T of exponent a is defined as the partition T^a having as terms $\frac{x_i}{\text{gcd}(a, x_i)}$ repeated $\text{gcd}(a, x_i)$ times for all $i \in \{1, \dots, r\}$. We say that T^a is a proper power of T if $[1^n] \neq T^a \neq T$. Note that T^a is a proper power of T if and only if $\text{gcd}(a, o(T)) \neq 1, o(T)$. For instance, $[2, 5, 6]^2 = [1^2, 3^2, 5]$ is a proper power of $[2, 5, 6] \in \mathcal{T}(13)$.

Let $\psi \in S_n$. The type of ψ is the partition of n given by the unordered list $T_\psi = [x_1, \dots, x_r]$ of the sizes x_i of the r orbits of ψ on N . In particular, $T_{id} = [1^n]$. The type map $t : S_n \rightarrow \mathcal{T}(n)$ which maps ψ into T_ψ is surjective and it is well known that $T_\psi = T_\varphi$ if and only if $\psi, \varphi \in S_n$ are conjugate in S_n . If $X \subseteq S_n$, then $t(X)$ is the set of types admissible for X in the sense of [3, Section 4.1], and is denoted by $\mathcal{T}(X)$. We also set $\mathcal{T}_0(X) = \mathcal{T}(X) \setminus \{[1^n]\}$. Note that $\mathcal{T}(S_n) = \mathcal{T}(n) \supsetneq \mathcal{T}(A_n)$ for all $n \geq 2$.

Given $G \leq S_n$, the *power type graph* $P(\mathcal{T}(G))$ is defined as the graph with vertex set $\mathcal{T}(G)$ and edge set $E_{\mathcal{T}(G)}$, where $\{T_1, T_2\} \in E_{\mathcal{T}(G)}$, for some $T_1, T_2 \in \mathcal{T}(G)$, if there exists $a \in \mathbb{N}$ such that $T_1 = T_2^a$ or $T_2 = T_1^a$. The *proper power type graph* $P_0(\mathcal{T}(G)) = (\mathcal{T}_0(G), E_{\mathcal{T}_0(G)})$ is defined as the $[1^n]$ -deleted

subgraph of $P(\mathcal{T}(G))$. Note that given $\psi, \varphi \in G_0$, there is a proper edge incident to $T_\psi, T_\varphi \in \mathcal{T}_0(G)$ in $P_0(\mathcal{T}(G))$ if and only if one of T_ψ, T_φ is a proper power of the other. We denote by $c_0(\mathcal{T}(G))$ the number of components of $P_0(\mathcal{T}(G))$. Recall that $\mathcal{O}_0(G)$ may be seen as a quotient of $P_0(\mathcal{T}(G))$ ([4, Proposition 5.6]).

For every $\psi \in S_n$, the type of $[\psi] \in [S_n]$ is defined by $T_{[\psi]} = T_\psi$. The map $\tilde{t}: [G]_0 \rightarrow \mathcal{T}_0(G)$ defined by $\tilde{t}([\psi]) = T_\psi$ for all $[\psi] \in [G]_0$, gives a complete 2-homomorphism from $P_0(\mathcal{T}(G))$ to $\tilde{P}_0(G)$ so that $P_0(\mathcal{T}(G))$ is a quotient of $\tilde{P}_0(G)$ ([4, Proposition 5.4]). For $X \subseteq S_n$ let $[X] = \{[x] \in [S_n] : x \in X\}$. Then, accordingly to [3, Section 4.1], $\tilde{t}([X]) = t(X) = \mathcal{T}(X)$ is the set of types admissible for $[X]$. If \hat{X} is a subgraph of $\tilde{P}_0(G)$ the set of types admissible for \hat{X} , denoted by $\mathcal{T}(\hat{X})$, is given by the set of types admissible for $V_{\hat{X}}$. In particular, for C component of $\tilde{P}_0(A_n)$ we have $\mathcal{T}(C) = \{T \in \mathcal{T}(G_0) : \text{there exists } [\psi] \in V_C \text{ with } T_\psi = T\}$.

In the fusion controlled case, \tilde{t} is also an orbit homomorphism ([4, Proposition 6.2]) and thus we can deduce the number of components of $\tilde{P}_0(G)$ from those of $P_0(\mathcal{T}(G))$ ([4, Theorem A]).

In [4], we proved that the graphs $P_0(G), \tilde{P}_0(G), P_0(\mathcal{T}(G)), \mathcal{O}_0(G)$ are very strictly related and we computed simultaneously the number of components of those graphs for $G = S_n$ ([4, Theorem B]). In particular we saw that, for $n \geq 8$, the components of $\tilde{P}_0(S_n), P_0(\mathcal{T}(S_n))$ and $\mathcal{O}_0(S_n)$, apart from one, are isolated vertices and that the 2-connectivity of $P(S_n)$ is equivalent to that of $P(\mathcal{T}(S_n))$ as well as to that of $\mathcal{O}(S_n)$, that is, to being $n = 2$ or none of n and $n - 1$ a prime ([4, Corollary C]). Here, we find analogous results about A_n applying the same general algorithmic procedure. Since, for $n \in \{1, 2\}$, $P_0(A_n)$ is the empty graph, we deal with the set of graphs

$$\mathcal{G} = \{P(A_n), \tilde{P}(A_n), P(\mathcal{T}(A_n)), \mathcal{O}(A_n)\}$$

and the set of their corresponding proper graphs

$$(2.2) \quad \mathcal{G}_0 = \{P_0(A_n), \tilde{P}_0(A_n), P_0(\mathcal{T}(A_n)), \mathcal{O}_0(A_n)\},$$

for $n \geq 3$. If $X \in \mathcal{G}$, we denote the corresponding proper graph in \mathcal{G}_0 with X_0 . Note that every $X \in \mathcal{G}$ is connected because it contains a vertex adjacent to any other vertex and it is 2-connected if and only if X_0 is connected. Let

$$\mathcal{C}_0(A_n), \tilde{\mathcal{C}}_0(A_n), \mathcal{C}_0(\mathcal{T}(A_n)), \mathcal{C}_0(\mathcal{O}(A_n))$$

be the sets of components of

$$P_0(A_n), \tilde{P}_0(A_n), P_0(\mathcal{T}(A_n)), \mathcal{O}_0(A_n)$$

respectively so that, due to the given notation, we have

$$c_0(A_n) = |\mathcal{C}_0(A_n)|, \tilde{c}_0(A_n) = |\tilde{\mathcal{C}}_0(A_n)|, c_0(\mathcal{T}(A_n)) = |\mathcal{C}_0(\mathcal{T}(A_n))|, c_0(\mathcal{O}(A_n)) = |\mathcal{C}_0(\mathcal{O}(A_n))|.$$

The following manageable inequalities link the number of components of the graphs in \mathcal{G}_0 . They follow immediately from [4, Propositions 5.4, 5.6] and [3, Proposition 3.2].

$$(2.3) \quad \forall n \geq 3, \quad c_0(\mathcal{O}(A_n)) \leq c_0(\mathcal{T}(A_n)) \leq \tilde{c}_0(A_n) = c_0(A_n).$$

In this paper, we find in one go the values of $c_0(A_n) = \tilde{c}_0(A_n), c_0(\mathcal{T}(A_n))$ and $c_0(\mathcal{O}(A_n))$ for all $n \geq 3$, and give information on the components of the graphs in \mathcal{G}_0 .

The idea of studying $P_0(G)$ through the proper quotient power graph or through the proper order graph is not a complete novelty. Even though in the literature it is not made explicit, it is tacitly implicated all the times in which the cyclic subgroups of G or its element orders are used as demonstration tools. For instance, in our language, the formula for computing the number of edges of $P_0(G)$ in [14, Corollary 5], which is an application of [7, Theorem 4.2], sounds as $|E_{P_0(G)}^*| = \frac{1}{2} \sum_{m \in \mathcal{O}_0(G)} s_m(2m - \phi(m) - 3)$, where s_m denotes the number of elements of order m in G and ϕ denotes the Euler totient function. It is instead completely original the idea of studying the proper power graph through the proper power type graph.

Let P denote the set of prime numbers. For $b, c \in \mathbb{N}$, put

$$bP + c = \{x \in \mathbb{N} : x = bp + c, \text{ for some } p \in P\}$$

and define

$$(2.4) \quad A = P \cup (P + 1) \cup (P + 2) \cup (2P) \cup (2P + 1).$$

The set of integers A plays the main role in our connectivity problem for the graphs in \mathcal{G}_0 .

Theorem A. *Let P be the set of prime numbers and A as in (2.4).*

The values of $c_0(A_n) = \tilde{c}_0(A_n)$, $c_0(\mathcal{T}(A_n))$ and $c_0(\mathcal{O}(A_n))$ are as follows.

(i) *For $3 \leq n \leq 10$, they are given in Table 1 below.*

TABLE 1. $c_0(A_n)$, $\tilde{c}_0(A_n)$, $c_0(\mathcal{T}(A_n))$ and $c_0(\mathcal{O}(A_n))$, for $3 \leq n \leq 10$.

| n | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-------------------------------|---|---|----|-----|-----|-----|------|-------|
| $c_0(A_n) = \tilde{c}_0(A_n)$ | 1 | 7 | 31 | 121 | 421 | 962 | 5442 | 29345 |
| $c_0(\mathcal{T}(A_n))$ | 1 | 2 | 3 | 4 | 4 | 3 | 4 | 3 |
| $c_0(\mathcal{O}(A_n))$ | 1 | 2 | 3 | 3 | 3 | 2 | 2 | 1 |

(ii) *For $n \geq 11$, they are given by Table 2 below.*

TABLE 2. $c_0(A_n)$, $\tilde{c}_0(A_n)$, $c_0(\mathcal{T}(A_n))$ and $c_0(\mathcal{O}(A_n))$ for $n \geq 11$.

| $n \geq 11$ | $c_0(A_n) = \tilde{c}_0(A_n)$ | $c_0(\mathcal{T}(A_n))$ | $c_0(\mathcal{O}(A_n))$ |
|---|--|-------------------------|-------------------------|
| $n - 2, \frac{n-1}{2} \in P$ | $\frac{n(n-1)(n-4)!}{2} + \frac{4n(n-2)(n-4)!}{n-1} + 1$ | 3 | 2 |
| $n, \frac{n-1}{2} \in P$ | $(n-2)! + \frac{4n(n-2)(n-4)!}{n-1} + 1$ | 3 | 2 |
| $n, n-2 \in P$ | $(n-2)! + \frac{n(n-1)(n-4)!}{2} + 1$ | 3 | 3 |
| $n-2 \in P, n \notin P, \frac{n-1}{2} \notin P$ | $\frac{n(n-1)(n-4)!}{2} + 1$ | 2 | 2 |
| $\frac{n-1}{2} \in P, n \notin P, n-2 \notin P$ | $\frac{4n(n-2)(n-4)!}{n-1} + 1$ | 2 | 1 |
| $n \in P, n-2 \notin P, \frac{n-1}{2} \notin P$ | $(n-2)! + 1$ | 2 | 2 |
| $n-1 \in P, \frac{n}{2} \notin P$ | $n(n-3)! + 1$ | 2 | 2 |
| $n-1, \frac{n}{2} \in P$ | $\frac{4(n-1)(n-3)!}{n} + n(n-3)! + 1$ | 3 | 2 |
| $\frac{n}{2} \in P, n-1 \notin P$ | $\frac{4(n-1)(n-3)!}{n} + 1$ | 2 | 1 |
| $n \notin A$ | 1 | 1 | 1 |

We emphasise that the proof of Theorem A is conducted similarly to the symmetric case in [4] but it is more tricky and requires some more technicalities and attention. The 2-connectivity of the proper power graph of a finite group G was studied by Doostabadi and Farrokhi in [8] for nilpotent groups, groups admitting a partition, symmetric and alternating groups. They used ingenious ad hoc arguments for the various classes of finite groups considered, but did not create a general procedure to determine $c_0(G)$. Unfortunately, the computation of $c_0(A_n)$ in [8, Theorem 4.7] is not generally correct, with mistakes related to infinitely many n . For instance, they claim that $c_0(A_{21}) = 19! + 1$, while we found that $c_0(A_{21}) = 21 \cdot 10 \cdot 17! + 1$. The more conspicuous error in [8] seems to be considering the case $n \in 2P + 2$ as significant, while Table 2 shows it is not.

Going beyond the mere counting, we describe the components of the proper power graph of A_n and their quotients. Note that for $n \notin A$, with $n \neq 3$, no $X_0 \in \mathcal{G}_0$ is a complete graph because X_0 admits as quotient $\mathcal{O}_0(A_n)$ which is surely incomplete having as vertices at least two primes.

Theorem B. *Let $n \in A$, with $n \geq 11$, and $\tilde{\Omega}_n$ be the component of $\tilde{P}_0(A_n)$ containing [(12)(34)].*

- (i) $\tilde{P}_0(A_n)$ consists of the main component $\tilde{\Omega}_n$ and of some isolated vertices of prime order, with at most two primes involved.
- (ii) $P_0(A_n)$ consists of the main component induced by $\{\psi \in A_n \setminus \{id\} : [\psi] \in V_{\tilde{\Omega}_n}\}$ and of some complete graphs on $p - 1$ vertices, with at most two primes p involved.
- (iii) $P_0(\mathcal{T}(A_n))$ consists of the main component $\tilde{t}(\tilde{\Omega}_n)$ and of some isolated vertices of prime order, with at most two primes involved.
- (iv) $\mathcal{O}_0(A_n)$ consists of the main component $\tilde{o}(\tilde{\Omega}_n)$ and of some isolated vertices which are primes, with at most two primes involved.

In all the above cases, the main component is never complete.

Complete information about the components of the graphs in \mathcal{G}_0 , for $3 \leq n \leq 10$ can be found within the proofs of Theorems 4.2, 5.5 and 6.2, taking into account Lemma 3.2 for $P_0(A_n)$. In particular, looking at the details, one can easily check that all the components of $X_0 \in \mathcal{G}_0$ apart from one are isolated vertices (complete graphs when $X_0 = P_0(A_n)$) if and only if $n \geq 11$ or $n = 3$.

Corollary C. *Let $n \geq 3$ and A as in (2.4). Then the following facts are equivalent:*

- (i) $P(A_n)$ is 2-connected;
- (ii) $P_0(A_n)$ is connected;
- (iii) $\tilde{P}_0(A_n)$ is connected;
- (iv) $P_0(\mathcal{T}(A_n))$ is connected;
- (v) $n = 3$ or $n \notin A$.

In particular, there exist infinitely many n such that $P(A_n)$ is 2-connected.

Remark. (a) $n = 16$ is the minimum n for which $P(A_n)$ is 2-connected with A_n non-abelian simple.

- (b) Exhibiting an infinite family of simple groups with 2-connected power graph, we give a positive answer to the question posed by Pourgholi, Yousefi-Azari and Ashrafi [15, Question 13].
- (c) Since Corollary C confirms the role of the proper power type graph in the comprehension of the 2-connectivity of the power graph for the alternating group, we wonder if an analogous construction is possible for any simple group.

As recalled before, for the symmetric group the 2-connectivity of the order graph is equivalent to the 2-connectivity of the power graph. This does not happen for the alternating group.

Corollary D. *Let $n \geq 3$. Then $\mathcal{O}(A_n)$ is 2-connected if and only if either $n = 3$ or none of $n, n - 1$ and $n - 2$ is a prime. The maximum number of components of $\mathcal{O}_0(A_n)$ is 3 and it is realised if and only if $n = 6$ or both n and $n - 2$ are prime.*

3. THE PROCEDURE FOR COMPUTING $c_0(A_n)$

In this section, we explain the terminology involved in the procedure which we are going to use for the computation of $c_0(A_n)$. We also present some results about the components of the graphs in \mathcal{G}_0 .

3.1. Permutations of a fixed type. Let $T = [m_1^{t_1}, \dots, m_k^{t_k}] \in \mathcal{T}(A_n)$. The number of permutations of type $T = [m_1^{t_1}, \dots, m_k^{t_k}]$ in A_n is the same as the number of this type in S_n and is given by

$$(3.1) \quad \mu_T(A_n) = \frac{n!}{m_1^{t_1} \dots m_k^{t_k} t_1! \dots t_k!}$$

while the number $\mu_T[A_n]$ of vertices of type T in $[A_n]$ coincides with $k_{\tilde{P}_0(A_n)(T)}$ and thus, by [4, Lemma 6.5]), is given by

$$(3.2) \quad \mu_T[A_n] = \frac{\mu_T(A_n)}{\phi(o(T))}.$$

3.2. Components of graphs in \mathcal{G}_0 . Let \mathcal{G}_0 be the set of graphs defined in (2.2). Since we do not want to reduce our research to a mere counting of components, we need to collect some information about the components of the graphs in \mathcal{G}_0 .

Proposition 3.1. *Let $\varphi : X \rightarrow Y$ be a homomorphism between the graphs X and Y .*

- (i) *If \hat{X} is a complete subgraph of X , then $\varphi(\hat{X})$ is a complete subgraph of Y .*
- (ii) *Let φ be pseudo-covering.*
 - (a) *If C is a component of X , then $\varphi(C)$ is a component of Y .*
 - (b) *For every vertex $x \in V_X$, $\varphi(C_X(x)) = C_Y(\varphi(x)) \cong C_X(x)/\sim_\varphi$.*

Proof. (i) is straightforward; (ii) comes from [3, Proposition 5.11]. \square

Lemma 3.2. *Let π be the projection of $P_0(A_n)$ on its quotient $\tilde{P}_0(A_n)$. Then, the map from $\mathcal{C}_0(A_n)$ to $\tilde{\mathcal{C}}_0(A_n)$ which associates to every $C \in \mathcal{C}_0(A_n)$, the component $\pi(C)$ is a bijection. Given $\tilde{C} \in \tilde{\mathcal{C}}_0(A_n)$, the set of vertices of the unique $C \in \mathcal{C}_0(A_n)$ such that $\pi(C) = \tilde{C}$ is given by $\pi^{-1}(V_{\tilde{C}})$.*

Proof. Apply [4, Lemma 3.7] to the group A_n . \square

The above lemma says that, once the components of $\tilde{P}_0(A_n)$ are known, it is immediate to recover those of $P_0(A_n)$. Moreover it leads to a useful result about isolated vertices in the graphs belonging to \mathcal{G}_0 .

Lemma 3.3.

- (i) *The type $T \in \mathcal{T}_0(A_n)$ is isolated in $P_0(\mathcal{T}(A_n))$ if and only if each $[\psi] \in [A_n]_0$ of type T is isolated in $\tilde{P}_0(A_n)$.*
- (ii) *If $m \in \mathcal{O}_0(A_n)$ is isolated in $\mathcal{O}_0(A_n)$, then each vertex of order m is isolated in $\tilde{P}_0(A_n)$.*
- (iii) *If, for some $\psi \in A_n$, $[\psi]$ is isolated in $\tilde{P}_0(A_n)$, then $o(\psi)$ is a prime p and the component of $P_0(A_n)$ containing ψ is a complete graph on $p-1$ vertices.*

Proof. (i) follows from [4, Proposition 6.3]; (ii) and (iii) follow from [4, Lemma 4.5]. \square

Let $C \in \tilde{\mathcal{C}}_0(A_n)$ and $T \in \mathcal{T}_0(A_n)$. Applying the definitions given in Section 2.2 to the graphs $X = \tilde{P}_0(A_n)$ and $Y = P_0(\mathcal{T}(A_n))$ and to the homomorphism \tilde{t} , we have that $k_C(T)$ is the multiplicity of T in C , that is, the number of vertices $[\psi] \in V_C$ such that $T_\psi = T$; $\tilde{\mathcal{C}}_0(A_n)_T$ is the set of components in $\tilde{\mathcal{C}}_0(A_n)$ which are admissible for T . Put $\tilde{c}_0(A_n)_T = |\tilde{\mathcal{C}}_0(A_n)_T|$. By [4, Lemma 6.4] we deduce immediately the following result.

Lemma 3.4. *Let $T \in \mathcal{T}_0(A_n)$ and $C \in \tilde{\mathcal{C}}_0(A_n)_T$. Then the following facts hold:*

- (i) $\mathcal{T}(C) = V_{C(T)}$;
- (ii) $\tilde{o}(C)$ is a connected subgraph of the component of $\mathcal{O}_0(A_n)$ containing $o(T)$.

From [4, Lemma 6.6], we know that

$$(3.3) \quad \tilde{c}_0(A_n)_T = \frac{\mu_T[A_n]}{k_C(T)}.$$

By [4, Theorem A] applied to the alternating group, we get the following reediting of the Procedure [3, 6.10] for counting the number of components of $\tilde{P}_0(A_n)$.

3.5. Procedure to compute $\tilde{c}_0(A_n)$

I) *Selection of types T_i and components C_i*

Start : Pick $T_1 \in \mathcal{T}_0(A_n)$ and choose any $C_1 \in \tilde{\mathcal{C}}_0(A_n)_{T_1}$.

Basic step : Given $T_1, \dots, T_i \in \mathcal{T}_0(A_n)$ and $C_1, \dots, C_i \in \tilde{\mathcal{C}}_0(A_n)$ such that $C_j \in \tilde{\mathcal{C}}_0(A_n)_{T_j}$ ($1 \leq j \leq i$), choose any $T_{i+1} \in \mathcal{T}_0(A_n) \setminus \bigcup_{j=1}^i \mathcal{T}(C_j)$ and any $C_{i+1} \in \tilde{\mathcal{C}}_0(A_n)_{T_{i+1}}$.

Stop : The procedure stops in $c_0(\mathcal{T}(A_n))$ steps.

II) *The value of $\tilde{c}_0(A_n)$*

Compute the integers

$$(3.4) \quad \tilde{c}_0(A_n)_{T_j} = \frac{\mu_{T_j}[A_n]}{k_{C_j}(T_j)} \quad (1 \leq j \leq c_0(\mathcal{T}(A_n)))$$

and sum them up to get $\tilde{c}_0(A_n)$.

The complete freedom in the choice of the $C_j \in \tilde{\mathcal{C}}_0(A_n)_{T_j}$ allows us to compute each $\tilde{c}_0(A_n)_{T_j} = \frac{\mu_{T_j}[A_n]}{k_{C_j}(T_j)}$, selecting C_j as the component containing $[\psi]$, for $[\psi]$ chosen as preferred among the $\psi \in A_n$ with $T_\psi = T_j$. We will apply this fact with no further mention. We emphasize also that the computing of $\mu_{T_j}[A_n]$ is made easy by (3.2) and (3.1). Remarkably, the number $c_0(\mathcal{T}(A_n))$ counts the steps of the procedure.

4. SOME SMALL DEGREES

In this section, we point out some general properties of the graphs in \mathcal{G}_0 and use them to determine $c_0(A_n) = \tilde{c}_0(A_n)$, $c_0(\mathcal{T}(A_n))$ and $c_0(\mathcal{O}(A_n))$, for $3 \leq n \leq 7$. Since $[A_3]_0 = \{[(1 \ 2 \ 3)]\}$, we trivially have $\tilde{c}_0(A_3) = c_0(\mathcal{T}(A_3)) = c_0(\mathcal{O}(A_3)) = 1$.

Lemma 4.1. *Let p be a prime number and $n \in \{p, p+1, p+2\}$, with $n \geq 4$. Then the following facts hold:*

- (i) $p \in \mathcal{O}_0(A_n)$ and is isolated in $\mathcal{O}_0(A_n)$. If $[\psi] \in [A_n]_0$ has order p , then $[\psi]$ is isolated in $\tilde{P}_0(A_n)$ and T_ψ is isolated in $P_0(\mathcal{T}(A_n))$;
- (ii) the number c_p of components of $\tilde{P}_0(A_n)$ containing elements of order p is given by the following table.

TABLE 3.

| n | c_p |
|----------------------|------------------------------|
| p | $(p-2)!$ |
| $p+1$ | $(p+1)(p-2)!$ |
| $p+2, p \text{ odd}$ | $\frac{(p+2)(p+1)(p-2)!}{2}$ |
| 4 | 3 |

Proof. (i) By the assumptions on n , we get $p \mid \frac{n!}{2}$ so that A_n admits elements of order p , but there exists no element in A_n with order kp for $k \geq 2$. This says that p is isolated in $\mathcal{O}_0(A_n)$. Then, by Lemma 3.3, $[\psi]$ is isolated in $\tilde{P}_0(A_n)$ and T_ψ is isolated in $P_0(\mathcal{T}(A_n))$, for all $[\psi] \in [A_n]$ with $o(\psi) = p$.

(ii)-(iv) We consider the elements of order p in A_n . If $n = p$, they are those of type $[p]$; if $n = p+1$, those of type $[1, p]$; if $n = p+2$, those of type $[1^2, p]$ if p is odd and those of type $[2^2]$ if $p = 2$. So the counting follows from (3.3) through (3.1) and (3.2). □

Theorem 4.2. *For $3 \leq n \leq 7$, the values of $c_0(A_n) = \tilde{c}_0(A_n)$, $c_0(\mathcal{T}(A_n))$ and $c_0(\mathcal{O}(A_n))$ are given by the following table.*

Proof. Let $G = A_n$, for $3 \leq n \leq 7$, acting on $N = \{1, \dots, n\}$. We compute $\tilde{c}_0(A_n)$ separately for each degree. Since $\tilde{P}_0(A_4)$ has only three vertices of type $[2^2]$ and four of type $[1, 3]$ and all of them are isolated, we have $\tilde{c}_0(A_4) = 7$, $c_0(\mathcal{T}(A_4)) = c_0(\mathcal{O}(A_4)) = 2$.

TABLE 4. $c_0(A_n) = \tilde{c}_0(A_n)$ and $c_0(\mathcal{T}(A_n))$, for $3 \leq n \leq 7$.

| n | 3 | 4 | 5 | 6 | 7 |
|-------------------------------|---|---|----|-----|-----|
| $c_0(A_n) = \tilde{c}_0(A_n)$ | 1 | 7 | 31 | 121 | 421 |
| $c_0(\mathcal{T}(A_n))$ | 1 | 2 | 3 | 4 | 4 |
| $c_0(\mathcal{O}(A_n))$ | 1 | 2 | 3 | 3 | 3 |

In $\tilde{P}_0(A_5)$, by Lemma 4.1, all the vertices of type $[1^2, 3]$ and $[5]$ are isolated. Moreover it is immediately checked that also the vertices of type $[1, 2^2]$ are isolated. In other words the graph $\tilde{P}_0(A_5)$ is totally disconnected. By (3.3) and (3.1), applying the procedure 3.5, we then get

$$\tilde{c}_0(A_5) = \mu_{[1, 2^2]}[A_5] + \mu_{[1^2, 3]}[A_5] + \mu_{[5]}[A_5] = 31,$$

while $c_0(\mathcal{T}(A_5)) = 3$. Since $\mathcal{O}_0(A_5) = \{2, 3, 5\}$ we also have $c_0(\mathcal{O}(A_5)) = 3$.

In $\tilde{P}_0(A_6)$, by Lemma 4.1, all the vertices of type $T_1 = [1, 5]$ are contained in $\tilde{c}_0(A_6)_{T_1} = 36$ components which are isolated vertices. Also, having in mind [4, Corollary 5.5], it is clear that all the vertices of types $T_2 = [1^3, 3]$ and $T_3 = [3^2]$ are isolated. Thus by (3.3) and (3.1), we get $\tilde{c}_0(A_6)_{T_2} = \tilde{c}_0(A_6)_{T_3} = 20$. Let $T_4 = [1^2, 2^2]$ and consider the component C_4 containing $\psi = (1\ 2)(3\ 4)$. Since T_4 admits no proper power, the edges incident to $[\psi]$ are given only by $\{[\psi], [\varphi]\}$ for those $\varphi \in A_6$ such that $\varphi^2 = \psi$. In particular $\varphi = (a\ b)(c\ d\ e\ f)$, where $\{a, b, c, d, e, f\} = N$ and $(c\ e)(d\ f) = (1\ 2)(3\ 4)$. It follows that $(a\ b) = (5\ 6)$, while either $(c\ d\ e\ f) = (1\ 3\ 2\ 4)$ or $(c\ d\ e\ f) = (3\ 1\ 4\ 2)$. On the other hand, by [4, Lemma 5.2], the type $T_\varphi = [2, 4]$ is not a proper power, because its terms are distinct. Thus C_4 is the path:

$$[(1\ 3\ 2\ 4)(5\ 6)], [(1\ 2)(3\ 4)], [(3\ 1\ 4\ 2)(5\ 6)]$$

and $k_{C_4}(T_4) = 1$. So, by (3.3), $\tilde{c}_0(A_6)_{T_4} = \mu_{[1^2, 2^2]}[A_6] = 45$. Since all the types admissible for A_6 have appeared, the application of the procedure 3.5 gives $\tilde{c}_0(A_6) = 121$, while $c_0(\mathcal{T}(A_6)) = 4$. Moreover $c_0(\mathcal{O}(A_6)) = 3$ and the components of $\mathcal{O}_0(A_6)$ have vertex sets $\{3\}$, $\{5\}$ and $\{2, 4\}$.

In $\tilde{P}_0(A_7)$, again by Lemma 4.1, all the vertices of type $T_1 = [1^2, 5]$ and $T_2 = [7]$ are isolated and $\tilde{c}_0(A_7)_{T_1} = 126$, $\tilde{c}_0(A_7)_{T_2} = 120$. Moreover, by [4, Corollary 5.5], it is clear that all the vertices of types $T_3 = [1, 3^2]$ are isolated and, by (3.3) and (3.1), we get $\tilde{c}_0(A_7)_{T_3} = 140$. Let C_i for $i \in \{1, 2, 3\}$ be a component admissible for T_i . Consider now the type $T_4 = [1^4, 3]$ and the component C_4 containing $\psi = (1\ 2\ 3)$. Since T_4 admits no proper power, the edges incident to $[\psi]$ are given only by $\{[\psi], [\varphi]\}$ for those $\varphi \in A_7$ such that $T_\varphi = [2^2, 3]$ and $\varphi^2 = \psi$. Then $\varphi = (a\ b)(c\ d)(e\ f\ g)$, where $\{a, b, c, d, e, f, g\} = N$ and $(e\ g\ f) = (1\ 2\ 3)$, $\{a, b, c, d\} = \{4, 5, 6, 7\}$. It follows that there are three choices for $[\varphi]$. Moreover each of those $[\varphi]$ is adjacent to exactly one vertex of type $[1^3, 2^2]$, which in turn is adjacent to three vertices of type $[1, 2, 4]$ and these elements are not adjacent to any other vertex. In particular we have $k_{C_4}(T_4) = 1$ and thus, by (3.3), $\tilde{c}_0(A_7)_{T_4} = \mu_{[1^4, 3]}[A_7] = 35$. Since $\bigcup_{i=1}^4 \mathcal{T}(C_i) = \mathcal{T}(A_7)$, the procedure 3.5 closes giving $\tilde{c}_0(A_7) = 421$ and $c_0(\mathcal{T}(A_7)) = 4$. Moreover $c_0(\mathcal{O}(A_7)) = 3$ and the three components of $\mathcal{O}_0(A_7)$ have vertex sets $\{2, 3, 4, 6\}$, $\{5\}$ and $\{7\}$. \square

5. THE DEGREES 8 AND 9

In this section, we determine $c_0(A_n) = \tilde{c}_0(A_n)$, $c_0(\mathcal{T}(A_n))$ and $c_0(\mathcal{O}(A_n))$ for $n \in \{8, 9\}$. We also obtain some results to treat all the cases with $n \geq 8$. For every $\psi \in A_n$, we denote by $M_\psi = \{i \in N : \psi(i) \neq i\}$ the support of ψ .

Definition 5.1. For $n \geq 8$, denote by $\tilde{\Omega}_n$ the component of $\tilde{P}_0(A_n)$ containing the vertex $[(12)(34)]$.

Lemma 5.2. Let $n \geq 8$.

- (i) All the vertices of $\tilde{P}_0(A_n)$ of type $[1^{n-4}, 2^2]$ belong to $\tilde{\Omega}_n$.
- (ii) $\mathcal{T}(\tilde{\Omega}_n) \supseteq \{[1^{n-4}, 2^2], [1^{n-3}, 3], [1^{n-7}, 2^2, 3]\}$.
- (iii) For every $T \in \mathcal{T}(\tilde{\Omega}_n)$, $\tilde{\Omega}_n$ contains all the vertices of $\tilde{P}_0(A_n)$ of type T and $\mathcal{T}(\tilde{\Omega}_n) = V_{C(T)}$.

Proof. (i) Let $[\pi_1]$ and $[\pi_2]$ be distinct vertices in $[A_n]_0$ of type $[1^{n-4}, 2^2]$. Then $\pi_1, \pi_2 \in A_n$ are distinct and share at most one transposition. Let $\pi_1 = (a b)(c d)$ for suitable distinct $a, b, c, d \in N$. We analyze the two possibilities:

- (1) π_1 and π_2 share a transposition;
- (2) π_1 and π_2 share no transposition.

(1) In this case we have $\pi_2 = (a b)(e f)$, for suitable distinct $e, f \in N$ and $\{c, d\} \neq \{e, f\}$. If $|\{c, d\} \cap \{e, f\}| = 1$, we may assume that $e = c$, that is, $\pi_2 = (a b)(c f)$. Since $n \geq 8$, there exist $x, y, z \in N \setminus \{a, b, c, e, f\}$ and we have the following path between $[\pi_1]$ and $[\pi_2]$:

$[\pi_1], [(a b)(c d)(x y z)], [(x y z)], [(a b)(c f)(x y z)], [\pi_2]$.

If $|\{c, d\} \cap \{e, f\}| = 0$, let $\pi = (a b)(c e)$. By the previous case, there is a path between $[\pi_1]$ and $[\pi]$ as well as a path between $[\pi_2]$ and $[\pi]$ and so also a path between $[\pi_1]$ and $[\pi_2]$.

(2) We distinguish the three possible subcases:

- (a) $|M_{\pi_1} \cap M_{\pi_2}| = 2$. Then $\pi_2 = (a e)(c f)$, for suitable $e, f \in N$, such that $\{a, e, c, f\} \cap \{a, b, c, d\} = \{a, c\}$. Let $\pi = (a b)(c f)$. Then, by 1), there exists a path between $[\pi_1]$ and $[\pi]$ and a path between $[\pi_2]$ and $[\pi]$. So there exists a path also between $[\pi_1]$ and $[\pi_2]$.
- (b) $|M_{\pi_1} \cap M_{\pi_2}| = 1$. Then $\pi_2 = (a e)(f g)$, for suitable $e, f, g \in N$ such that $\{a, e, f, g\} \cap \{a, b, c, d\} = \{a\}$. Now consider $\pi = (a b)(f g)$ and argue as in (a).
- (c) $|M_{\pi_1} \cap M_{\pi_2}| = 0$. Then $\pi_2 = (e f)(g h)$ for suitable $e, f, g, h \in N$ such that $\{e, f, g, h\} \cap \{a, b, c, d\} = \emptyset$. Now consider $\pi = (a b)(e f)$ and argue as in (a).

This shows that all the vertices of $\tilde{P}_0(A_n)$ of type $[1^{n-4}, 2^2]$ lie in the same component $\tilde{\Omega}_n$.

(ii) Collecting the types met in the paths used for i), we immediately get

$$\mathcal{T}(\tilde{\Omega}_n) \supseteq \{[1^{n-4}, 2^2], [1^{n-3}, 3], [1^{n-7}, 2^2, 3]\}.$$

(iii) Let $T \in \mathcal{T}(\tilde{\Omega}_n)$. The fact that $\mathcal{T}(\tilde{\Omega}_n) = V_{C(T)}$ is an application of Lemma 3.4. The fact that $\tilde{\Omega}_n$ contains all the vertices in $[A_n]_0$ of type T is instead a consequence of (i) and of [4, Corollary 6.7(iii)]. \square

Corollary 5.3. *Let $n \geq 8$ and Ω_n be the unique component of $P_0(A_n)$ such that $\pi(\Omega_n) = \tilde{\Omega}_n$. Then neither one of the components $\Omega_n, \tilde{\Omega}_n, \tilde{t}(\tilde{\Omega}_n)$ of the graphs $P_0(A_n), \tilde{P}_0(A_n), P_0(\mathcal{T}(A_n))$ respectively, nor the connected subgraph $\tilde{o}(\tilde{\Omega}_n)$ of $\mathcal{O}_0(A_n)$ is a complete graph.*

Proof. First of all note that the existence of a unique component Ω_n of $P_0(A_n)$ such that $\pi(\Omega_n) = \tilde{\Omega}_n$ is guaranteed by Lemma 3.2. Moreover, by Proposition 3.1, we have that $\tilde{t}(\tilde{\Omega}_n)$ is a component of $\tilde{P}_0(A_n)$ with $V_{\tilde{t}(\tilde{\Omega}_n)} = \mathcal{T}(\tilde{\Omega}_n)$. On the other hand, by [4, Proposition 5.6], the map $o_{\mathcal{T}} : \mathcal{T}_0(A_n) \rightarrow \mathcal{O}_0(A_n)$ defined by $o_{\mathcal{T}}(T) = o(T)$ for all $T \in \mathcal{T}_0(A_n)$, defines a complete 2-homomorphism $o_{\mathcal{T}} : P_0(\mathcal{T}(A_n)) \rightarrow \mathcal{O}_0(A_n)$ such that $o_{\mathcal{T}} \circ \tilde{t} = \tilde{o}$, which gives $\tilde{o}(\tilde{\Omega}_n) = o_{\mathcal{T}}(\tilde{t}(\tilde{\Omega}_n))$. It follows that we can interpret the sequence of graphs

$$\Omega_n, \tilde{\Omega}_n, \tilde{t}(\tilde{\Omega}_n), \tilde{o}(\tilde{\Omega}_n)$$

as

$$(5.1) \quad \Omega_n, \pi(\Omega_n), \tilde{t}(\pi(\Omega_n)), o_{\mathcal{T}}(\tilde{t}(\pi(\Omega_n))).$$

It is immediate to check that $\tilde{o}(\tilde{\Omega}_n)$ is not a complete graph because, by Lemma 5.2, it admits as vertices the integer 2 and 3 and no edge exists between them in $\mathcal{O}_0(A_n)$. Then to deduce that no graph in the sequence (5.1) is complete, we start from the bottom and apply three times Proposition 3.1 (i). \square

Note that, in principle, $\tilde{o}(\tilde{\Omega}_n)$ is not a component of $\mathcal{O}_0(A_n)$ because \tilde{o} is not, in general, pseudo-covering. For instance $\tilde{o}(\tilde{\Omega}_9)$ is not a component of $\mathcal{O}_0(A_9)$ because 3 is a vertex of $\tilde{o}(\tilde{\Omega}_9)$ but 9, adjacent to 3 in $\mathcal{O}_0(A_9)$, is not a vertex of $\tilde{o}(\tilde{\Omega}_9)$. That fact is easily understood looking inside the proof of Theorem 5.5. Anyway we will see, along the proof of Theorem A, that $\tilde{o}(\tilde{\Omega}_n)$ is indeed a component at least for $n \geq 11$.

Lemma 5.4. *Let $n \in \{8, 9\}$.*

- (i) The vertices of $\tilde{P}_0(A_n)$ of type $[1^{n-8}, 2^4]$ belong to the same component $\tilde{\Lambda}_n$ of $\tilde{P}_0(A_n)$.
- (ii) $\mathcal{T}(\tilde{\Lambda}_n) = \{[1^{n-8}, 2^4], [1^{n-8}, 2, 6], [1^{n-6}, 3^2], [1^{n-8}, 4^2]\}$.
- (iii) $\tilde{\Lambda}_n \neq \tilde{\Omega}_n$.

Proof. Let first $n = 8$. Let $[\pi_1]$ and $[\pi_2]$ be distinct elements in $[A_8]_0$ of type $[2^4]$, with $\pi_1 = (a\ b)(c\ d)(e\ f)(g\ h)$, where $\{a, b, c, d, e, f, g, h\} = N$. Since π_1, π_2 are distinct, they share at most two transpositions. We analyze the three possibilities:

- (1) π_1 and π_2 share one transposition;
- (2) π_1 and π_2 share two transposition;
- (3) π_1 and π_2 share no transposition.

(1) In this case, since the 2-cycles in which π_1 splits commute and also the entries in each cycle commute, we can assume that $\pi_2 = (a\ b)(c\ e)(h\ f)(g\ d)$ and look to the path

$$[\pi_1], [(a\ b)(c\ e\ g\ d\ f\ h)], [(c\ g\ f)(e\ d\ h)], [(c\ h\ g\ e\ f\ d)(a\ b)], [\pi_2].$$

(2) Here, for the same considerations as in (1), we reduce to

$\pi_2 = (a\ b)(c\ d)(e\ g)(f\ h)$. Consider $\pi = (a\ b)(c\ e)(d\ h)(f\ g)$. Then by (1), there exists a path between $[\pi_1]$ and $[\pi]$ and a path between $[\pi_2]$ and $[\pi]$. Thus there exists also a path between $[\pi_1]$ and $[\pi_2]$.

(3) Here we reduce to the two cases $\pi_2 = (a\ c)(b\ d)(e\ g)(f\ h)$ or $\pi_2 = (a\ c)(b\ g)(e\ d)(f\ h)$. In the first case let $\pi = (a\ b)(c\ d)(e\ g)(f\ h)$. Then by (2), there exists a path between $[\pi_1]$ and $[\pi]$ and a path between $[\pi_2]$ and $[\pi]$. In the second case let $\pi = (a\ b)(c\ g)(e\ d)(f\ h)$. Then by (1), there exists a path between $[\pi_1]$ and $[\pi]$ and by (2), there exists a path between $[\pi_2]$ and $[\pi]$. In both cases we then get a path between $[\pi_1]$ and $[\pi_2]$.

This shows that all vertices of type $[2^4]$ in $[A_8]_0$ are in the same component $\tilde{\Lambda}_8$ of $\tilde{P}_0(A_8)$. Collecting the types met along the considered paths we also get $\mathcal{T}(\tilde{\Lambda}_8) \supseteq \{[2^4], [2, 6], [1^2, 3^2]\}$. It is also clear that $[4^2] \in \mathcal{T}(\tilde{\Lambda}_8)$ because $[(1\ 2\ 3\ 4)(5\ 6\ 7\ 8)]^2 = (1\ 3)(2\ 4)(5\ 7)(6\ 8)$. To show that $\mathcal{T}(\tilde{\Lambda}_8) = \{[2^4], [2, 6], [1^2, 3^2], [4^2]\}$ it is enough to observe that the set of types $\{[2^4], [2, 6], [1^2, 3^2], [4^2]\}$ is closed by proper powers within the types admissible for A_8 . Finally, since the type $[1^4, 2^2]$ is admissible for $\tilde{\Omega}_8$ but not for $\tilde{\Lambda}_8$ we deduce that $\tilde{\Lambda}_8 \neq \tilde{\Omega}_8$.

Next let $n = 9$ and let $[\pi_1], [\pi_2]$ be distinct elements in $[A_9]_0$ of type $[1, 2^4]$. If π_1, π_2 have the same fixed point we are inside a copy of A_8 and so there is a path between $[\pi_1]$ and $[\pi_2]$ in $\tilde{P}_0(A_8)$, which is also a path in $\tilde{P}_0(A_9)$. If π_1, π_2 have a different fixed point, without loss of generality we may assume that π_1 fixes 9 and π_2 fixes 8. Now consider $\varphi_1 = (1\ 2)(3\ 4)(5\ 6)(7\ 8)$ which fixes 9 and $\varphi_2 = (1\ 2)(3\ 4)(5\ 6)(7\ 9)$ which fixes 8. We have the following path between $[\varphi_1]$ and $[\varphi_2]$:

$$[\varphi_1], [(1\ 3\ 5\ 2\ 4\ 6)(7\ 8)], [(1\ 5\ 4)(3\ 2\ 6)], [(1\ 3\ 5\ 2\ 4\ 6)(7\ 9)], [\varphi_2].$$

Since π_i and φ_i have the same fixed point, there is a path between $[\pi_i]$ and $[\varphi_i]$, for all $i = 1, 2$ and thus also a path between $[\pi_1]$ and $[\pi_2]$. This shows that all the vertices of type $[1, 2^4]$ in $[A_9]_0$ are in the same component $\tilde{\Lambda}_9$ of $\tilde{P}_0(A_9)$. Collecting the types met along the paths we get $\mathcal{T}(\tilde{\Lambda}_9) \supseteq \{[1, 2^4], [1, 2, 6], [1^3, 3^2]\}$. It is also clear that $[1, 4^2] \in \mathcal{T}(\tilde{\Lambda}_9)$ because $[(1\ 2\ 3\ 4)(5\ 6\ 7\ 8)]^2 = (1\ 3)(2\ 4)(5\ 7)(6\ 8)$. Arguing as for $n = 8$, we then get

$$\mathcal{T}(\tilde{\Lambda}_9) = \{[1, 2^4], [1, 2, 6], [1^3, 3^2], [1, 4^2]\}.$$

Finally, since the type $[1^5, 2^2]$ is admissible for $\tilde{\Omega}_9$ but not for $\tilde{\Lambda}_9$ we deduce that $\tilde{\Lambda}_9 \neq \tilde{\Omega}_9$. □

Theorem 5.5. *Let $n \in \{8, 9\}$. Then the values of $c_0(A_n) = \tilde{c}_0(A_n)$, $c_0(\mathcal{T}(A_n))$ and $c_0(\mathcal{O}(A_n))$ are given in Table 5 below.*

Proof. Let $n = 8$. By Lemma 4.1, the prime 7 is isolated in $\mathcal{O}_0(A_8)$. It follows that also $T_1 = [1, 7]$ is isolated in $\tilde{P}_0(A_8)$ and thus $\tilde{c}_0(A_8)_{T_1} = 960$. By Lemma 5.2, all the vertices of $\tilde{P}_0(A_8)$ of type $T_2 = [1^4, 2^2]$ are in $\tilde{\Omega}_8$, so that $\tilde{c}_0(A_8)_{T_2} = 1$, and $\mathcal{T}(\tilde{\Omega}_8) \supseteq \{[1^4, 2^2], [1^5, 3], [1, 2^2, 3]\}$. Moreover looking at the

TABLE 5. $c_0(A_n) = \tilde{c}_0(A_n)$, $c_0(\mathcal{T}(A_n))$ and $c_0(\mathcal{O}(A_n))$, for $n = 8, 9$.

| n | 8 | 9 |
|-------------------------------|-----|------|
| $c_0(A_n) = \tilde{c}_0(A_n)$ | 962 | 5442 |
| $c_0(\mathcal{T}(A_n))$ | 3 | 4 |
| $c_0(\mathcal{O}(A_n))$ | 2 | 2 |

types admissible for A_8 which can have as proper power one of the types in $\{[1^4, 2^2], [1^5, 3], [1, 2^2, 3]\}$, we get immediately that

$$\mathcal{T}(\tilde{\Omega}_8) = \{[1^4, 2^2], [1^5, 3], [1, 2^2, 3], [3, 5], [1^3, 5], [1^2, 2, 4]\}.$$

Thus

$$V_{\tilde{\mathcal{O}}(\tilde{\Omega}_8)} = \{2, 3, 4, 5, 6, 15\},$$

so that $\tilde{\mathcal{O}}(\tilde{\Omega}_8)$ is included in the component of $\mathcal{O}_0(A_8)$ containing 2.

By Lemma 5.4, all the vertices of type $T_3 = [2^4]$ are in $\tilde{\Lambda}_8$ and

$$\mathcal{T}(\tilde{\Lambda}_8) = \{[2^4], [2, 6], [1^2, 3^2], [4^2]\}, \quad \tilde{\mathcal{O}}(\tilde{\Lambda}_8) = \{2, 3, 4, 6\}$$

so that $\tilde{c}_0(A_8)_{T_3} = 1$ and the procedure 3.5 closes giving $c_0(A_8) = 962$. The discussion above shows also that $c_0(\mathcal{T}(A_8)) = 3$ and $c_0(\mathcal{O}(A_8)) = 2$.

Let $n = 9$. By Lemma 4.1, the prime 7 is isolated in $\mathcal{O}_0(A_9)$. Thus, for $T_1 = [1^2, 7]$ we have $\tilde{c}_0(A_9)_{T_1} = 4320$. By Lemma 5.2, all the vertices of type $T_2 = [1^5, 2^2]$ are in $\tilde{\Omega}_9$ and $\mathcal{T}(\tilde{\Omega}_9) \supseteq \{[1^5, 2^2], [1^6, 3], [1^2, 2^2, 3]\}$. In particular $\tilde{c}_0(A_9)_{T_2} = 1$. Moreover, looking at the types admissible for A_9 which can have as proper power one of the types in $\{[1^5, 2^2], [1^6, 3], [1^2, 2^2, 3]\}$, we get that

$$\mathcal{T}(\tilde{\Omega}_9) = \{[1^5, 2^2], [1^6, 3], [1^2, 2^2, 3], [1, 3, 5], [1^4, 5], [1^3, 2, 4]\}$$

and so

$$V_{\tilde{\mathcal{O}}(\tilde{\Omega}_9)} = \{2, 3, 4, 5, 6, 15\}.$$

By Lemma 5.4, all the vertices of type $T_3 = [1, 2^4]$ are in $\tilde{\Lambda}_9$ and

$$\mathcal{T}(\tilde{\Lambda}_9) = \{[1, 2^4], [1, 2, 6], [1^3, 3^2], [1, 4^2]\}, \quad V_{\tilde{\mathcal{O}}(\tilde{\Lambda}_9)} = \{2, 3, 4, 6\}$$

so that $\tilde{c}_0(A_9)_{T_3} = 1$.

Finally consider the type $T_4 = [3^3]$. Let $\varphi \in A_9$ with $T_\varphi = T_4$ and let C_4 be the component of $\tilde{P}_0(A_9)$ containing $[\varphi]$. We show that $k_{C_4}(T_4) = 1$. Note, first of all, that T_4 has no proper power and that it is the proper power only of $[9]$. Moreover, by [4, Lemma 5.2], the type $[9]$ is not a proper power and its only proper power has exponent $a \in \mathbb{N}$, with $1 < a < 9$ and a not coprime to 9, so that $a = 3$. Thus, by Lemma 3.4, we have $V_{C_4}(T_4) = \{T_4, [9]\} = \mathcal{T}(C_4)$ and so also $V_{\tilde{\mathcal{O}}(C_4)} = \{3, 9\}$.

We show that $[\varphi]$ is the only vertex of type T_4 in C_4 , showing that there exists no path between $[\varphi]$ and some $[\sigma] \in [A_9]_0$ such that $[\sigma] \neq [\varphi]$ and $T_\sigma = T_4$. Let γ be a path of length at least 1 with end vertex $[\varphi]$. Since the only types admissible for C_4 are T_4 and $[9]$ and in a quotient power graph there exists no proper edge incident to vertices having the same type, we have that the first proper edge of γ is $\{[\varphi], [\psi]\}$, for some $\psi \in A_9$ such that $T_\psi = [9]$. Thus, by (2.1), $\{\varphi, \psi\}$ is a proper edge in $P_0(A_9)$ and so necessarily $\psi^3 = \varphi$. Consider now the proper edges of $\tilde{P}_0(A_9)$ of type $\{[\psi], [\delta]\}$, for some $[\delta] \in [A_9]_0$. Again, since the only types admissible for C_4 are T_4 and $[9]$, we get $T_\delta = T_4$. Thus, by (2.1), ψ and δ are one the power of the other. It follows that $\delta = \psi^3 = \varphi$. This means that γ is a path of length 1 in which one end is not of type T_4 . Then $\tilde{c}_0(A_9)_{T_4} = \mu_{T_4}[A_9] = 1120$. Since all the types admissible for A_9 have appeared, the procedure 3.5 stops giving $\tilde{c}_0(A_9) = 5442$. The discussion above shows also that $c_0(\mathcal{T}(A_9)) = 4$ and that $c_0(\mathcal{O}(A_9)) = 2$. □

Corollary 5.6. *Let $n \in \mathbb{N}$ with $4 \leq n \leq 9$. Then the vertices of even order in $\tilde{P}_0(A_n)$ are distributed in more than one component.*

Proof. For $4 \leq n \leq 7$, this is immediate by the case by case analysis of the proof of Theorem 4.2. For $8 \leq n \leq 9$, it follows by the case by case analysis of the proof of Theorem 5.5. \square

6. HIGHER DEGREES

In this section, we determine $c_0(A_n) = \tilde{c}_0(A_n)$, $c_0(\mathcal{T}(A_n))$ and $c_0(\mathcal{O}(A_n))$ for $n \geq 10$. We start showing that the fact that the vertices of even order are distributed in more than one component is a peculiarity of $4 \leq n \leq 9$.

Lemma 6.1. *For every $n \geq 10$, the vertices of even order of $\tilde{P}_0(A_n)$ are contained in $\tilde{\Omega}_n$.*

Proof. By Lemma 5.2, all the vertices of type $[1^{n-4}, 2^2]$ are contained in $\tilde{\Omega}_n$ of $\tilde{P}_0(A_n)$. Let $[\pi] \in [A_n]_0$ with $o(\pi) = 2k$, for a positive integer k . Then $o(\pi^k) = 2$, so that π^k is the product of s cycles of length two, for some $s \in \mathbb{N}$ even. If $s = 2$, then $[\pi^k] \in \tilde{\Omega}_n$ and so $[\pi] \in \tilde{\Omega}_n$. If $s \geq 4$ then $\pi^k = (a\ b)(c\ d)(e\ f)(g\ h)\sigma$, for suitable distinct $a, b, c, d, e, f, g, h \in N$ and $\sigma \in A_n$, with $\sigma = id$ or the product of $s - 4$ cycles of order two. Let $\varphi = (a\ c\ e\ b\ d\ f)(g\ h)\sigma$. Then $\varphi \in A_n$ and $\varphi^3 = \pi^k$. Due to $n \geq 10$, there exist distinct elements $i, j \in N \setminus \{a, b, c, d, e, f, g, h\}$ and we have the following path:

$$[\pi], [\pi^k], [\varphi], [(a\ e\ d)(c\ b\ f)], [(a\ e\ d)(c\ b\ f)(g\ h)(i\ j)], [(g\ h)(i\ j)].$$

Since $[(g\ h)(i\ j)] \in \tilde{\Omega}_n$, we also get $[\pi] \in \tilde{\Omega}_n$. \square

Theorem 6.2. $c_0(A_{10}) = \tilde{c}_0(A_{10}) = 29345$, $c_0(\mathcal{T}(A_{10})) = 3$ and $c_0(\mathcal{O}(A_{10})) = 1$.

Proof. By Lemma 6.1, the vertices of even order in $[A_{10}]_0$ are contained in $\tilde{\Omega}_{10}$. Thus, for $T_1 = [1^6, 2^2]$, we have $\tilde{c}_0(A_{10})_{T_1} = 1$. Moreover from the paths:

$$[(1\ 2\ 3)(4\ 5\ 6)], [(1\ 2\ 3)(4\ 5\ 6)(7\ 8)(9\ 10)],$$

and

$$[(1\ 2\ 3\ 4\ 5\ 6\ 7)], [(1\ 2\ 3\ 4\ 5\ 6\ 7)(8\ 9\ 10)], [(8\ 9\ 10)], [(8\ 9\ 10)(1\ 2\ 3\ 4\ 5)],$$

$$[(1\ 2\ 3\ 4\ 5)], [(1\ 2\ 3\ 4\ 5)(6\ 7)(8\ 9)]$$

we deduce that the types $[1^4, 3^2]$, $[1^3, 7]$, $[3, 7]$, $[1^7, 3]$, $[1^2, 3, 5]$ and $[1^5, 5]$ are admissible for $\tilde{\Omega}_{10}$. Since $[10] \notin \mathcal{T}(A_{10})$, the vertices of type $T_2 = [5^2] \in \mathcal{T}(A_{10})$ are instead isolated and thus $\tilde{c}_0(A_{10})_{T_2} = \mu_{[5^2]}[A_{10}] = 18144$. The same argument used for $\tilde{P}_0(A_9)$ shows also that for $T_3 = [1, 3^3]$ we have $\tilde{c}_0(A_{10})_{T_3} = \mu_{[1, 3^3]}[A_{10}] = 11200$. Since all the types admissible for A_{10} have appeared, the procedure 3.5 stops giving $\tilde{c}_0(A_{10}) = 29345$ and $c_0(\mathcal{T}(A_{10})) = 3$. Moreover, since $\tilde{o}(\tilde{\Omega}_{10})$ contains all the prime less than 10, we deduce that $c_0(\mathcal{O}(A_{10})) = 1$. \square

Lemma 6.3. *Let $n \geq 11$. Then the vertices of order 3 in $\tilde{P}_0(A_n)$ are contained in $\tilde{\Omega}_n$.*

Proof. Let $n \geq 11$ and $[\pi] \in [A_n]_0$, with $o(\pi) = 3$. Then π is the product of $s \geq 1$ cycles of length 3. To show that $[\pi] \in \tilde{\Omega}_n$, we show that whatever s is, there exists $\varphi \in A_n$ such that $\varphi^2 = \pi$ with $o(\varphi)$ even and apply Lemma 6.1. If $s = 1$ then, since $n \geq 11$, there exist $a, b, c, d \in N \setminus M_\pi$ and we take $\varphi = \pi^2(a\ b)(c\ d)$. Let $s = 2$ and $\pi = (a\ b\ c)(d\ e\ f)$. Then we take $\varphi = (a\ d\ b\ e\ c\ f)(g\ h)$, where $g, h \in N \setminus M_\pi$. Let $s = 3$ and $\pi = (a\ b\ c)(d\ e\ f)(g\ h\ i)$. Then we take $\varphi = (a\ d\ b\ e\ c\ f)(g\ i\ h)(j\ k)$, where $j, k \in N \setminus M_\pi$. Finally, let $s \geq 4$. Then $n \geq 12$ and $\pi = (a\ b\ c)(d\ e\ f)(g\ h\ i)(j\ k\ l)\sigma$, for suitable $a, b, c, d, e, f, g, h, i, j, k, l \in N$ and $\sigma \in A_n$, with $\sigma^3 = id$. We then take $\varphi = (a\ d\ b\ e\ c\ f)(g\ j\ h\ k\ i\ l)\sigma^2$. \square

Lemma 6.4. *Let $n \geq 11$ and p be a prime number such that $5 \leq p \leq n - 3$ and $p \neq \frac{n}{2}, \frac{n-1}{2}$. Then the vertices of order p in $\tilde{P}_0(A_n)$ are contained in $\tilde{\Omega}_n$.*

Proof. Let $[\pi] \in [A_n]_0$, with $o(\pi) = p$. Then $T_\pi \in \mathcal{T}$, where

$$\mathcal{T} = \{T_s = [1^{n-sp}, p^s] : s \in \mathbb{N}, s \leq n/p\}.$$

By Lemma 5.2 iii) it is enough to show that $\mathcal{T} \subseteq \mathcal{T}(\tilde{\Omega}_n)$. Recall now that, by Lemma 3.4, $\mathcal{T}(\tilde{\Omega}_n)$ is the set of vertices of a component of $P_0(\mathcal{T}(A_n))$. Thus, by Lemma 6.1 and Lemma 6.3, it is enough to prove that for all $s \in \mathbb{N}$, with $s \leq n/p$, there exist $T^* \in \mathcal{T}(A_n)$ with $o(T^*)$ even or $o(T^*) = 3$ and a path in $P_0(\mathcal{T}(A_n))$ between T^* and T_s . If $sp \leq n - 3$, consider $T = [p^s, 3, 1^{n-sp-3}]$. Since $T^3 = T_s$, we take $T^* = T^p = [3, 1^{n-3}]$. If $sp \geq n - 2$, since $p \leq n - 3$, we have $s \geq 2$. Let first $s = 2$. Then $n - 2 \leq 2p \leq n$ and, by $p \neq \frac{n}{2}, \frac{n-1}{2}$, we get $2p = n - 2$. So we can consider $T^* = [2p, 2]$, because $o(T^*)$ is even and $(T^*)^2 = T_2$. Next let $s = 3$ and consider $T = [3p, 1^{n-3p}]$; as $p \geq 5$, we have $T^3 = T_3$ while $o(T^p) = 3$ and we take $T^* = T^p$. Finally, for $s \geq 4$, we take $T^* = [(2p)^2, p^{s-4}, 1^{n-sp}]$ after having observed that $(T^*)^2 = T_s$ and that $o(T^*)$ is even. \square

Lemma 6.5. *Let $n \geq 11$ with $n = 2p$ or $n = 2p + 1$, for some prime p . Let $[\pi] \in [A_n]_0$, with $o(\pi) = p$. Then, the following holds:*

- (i) *if $T_\pi = [1^{n-2p}, p^2]$, then $[\pi]$ is an isolated vertex of $\tilde{P}_0(A_n)$;*
- (ii) *if $T_\pi = [1^{n-p}, p]$, then $[\pi] \in \tilde{\Omega}_n$;*
- (iii) *$p \in V_{\tilde{o}(\tilde{\Omega}_n)}$.*

Proof. If $T_\pi = [1^{n-2p}, p^2]$, since $n - 2p \leq 1$, T_π admits no proper power and is not a proper power of a type admissible for A_n . Thus, by Lemma 3.3, $[\pi]$ is isolated in $\tilde{P}_0(A_n)$. Next let $T_\pi = [1^{n-p}, p]$. By $n \geq 11$, we get $n - p \geq p \geq 5$ and thus, the type $T = [1^{n-p-3}, 3, p] \in \mathcal{T}(A_n)$ satisfies $o(T^p) = 3$, while $T^3 = T_\pi$. So, by Lemma 6.3, $T^p \in \mathcal{T}(\tilde{\Omega}_n)$ and, by Lemma 3.4, $T_\pi \in \mathcal{T}(\tilde{\Omega}_n)$. Finally, Lemma 5.2 iii), guarantees $[\pi] \in \tilde{\Omega}_n$. In particular, $p = \tilde{o}([\pi]) \in V_{\tilde{o}(\tilde{\Omega}_n)}$. \square

Lemma 6.6. *For $n \geq 11$, all the vertices of $\tilde{P}_0(A_n)$ apart those of order a prime p such that $p \geq n - 2$ or $p = \frac{n}{2}, \frac{n-1}{2}$ are contained in $\tilde{\Omega}_n$.*

Proof. Let $[\pi] \in [A_n]_0$, with $o(\pi)$ not a prime p such that $p \geq n - 2$ and not a prime p such that $p = \frac{n}{2}, \frac{n-1}{2}$. By Lemma 6.1, we can assume that $o(\pi)$ is odd. Let first $o(\pi) = p$, for some prime p . If $p = 3$, then Lemma 6.3 applies. If $p \geq 5$, then, since $p \leq n - 3$, Lemma 6.4 applies. Assume now that $o(\pi)$ is composite. If $o(\pi) = 3^k$ for some $k \in \mathbb{N}, k \geq 2$, then we have $o(\pi^{3^{k-1}}) = 3$ and, by Lemma 6.3, we get $[\pi^{3^{k-1}}] \in \tilde{\Omega}_n$, so that also $[\pi] \in \tilde{\Omega}_n$. Finally let $o(\pi) = tpq$, with $p \geq 3, q \geq 5$ prime numbers and t an odd positive integer. Since $o(\pi^t) = pq$, then T_π contains a term equal to pq or two terms equal to p and q respectively. In any case, as $n \geq 11$, this implies $o(\pi^{tp}) = q \leq n - 3$ so that, by Lemma 6.4, $[\pi^{tp}] \in \tilde{\Omega}_n$ and hence also $[\pi] \in \tilde{\Omega}_n$. \square

Proof of Theorem A . (i) Combine Theorems 4.2, 5.5 and 6.2.

(ii) Let $n \geq 11$ be fixed and recall that $c_0(A_n) = \tilde{c}_0(A_n)$. First note that, as $P \cap O_0(A_n) = \{p \in P : p \leq n\}$, we can reformulate Lemma 6.6 saying that all the vertices of $\tilde{P}_0(A_n)$ of order not belonging to

$$B(n) = P \cap \left\{n, n-1, n-2, \frac{n}{2}, \frac{n-1}{2}\right\}$$

are in $\tilde{\Omega}_n$. We refer to the numbers in $B(n)$ as to the critical orders for n . By Lemma 3.4 ii), there exists a unique component Θ_n of $\mathcal{O}_0(A_n)$ containing the connected subgraph $\tilde{o}(\tilde{\Omega}_n)$, and we have $V_{\Theta_n} \supseteq V_{\tilde{o}(\tilde{\Omega}_n)} \supseteq O_0(A_n) \setminus B(n)$.

If $n \notin A$, then $B(n) = \emptyset$ and thus all the vertices in $[A_n]_0$ belongs to $\tilde{\Omega}_n$. So, by (2.3), $c_0(A_n) = \tilde{c}_0(A_n) = c_0(\mathcal{T}(A_n)) = c_0(\mathcal{O}(A_n)) = 1$.

Next let $n \in A$. We examine all the possible positions of n with respect to the sets $P, P+1, P+2, 2P, 2P+1$ whose union is A . First of all we show that $P \cap (P+2) \cap (2P+1) = \emptyset$, proving that if $n \in P \cap (P+2)$, then $\frac{n-1}{2} \notin P$. Let $n = p \in P$ and $n - 2 = p - 2 = q \in P$. Then, due to $n \geq 11$, we have $p, q \geq 11$. Since 3 divides $n(n-1)(n-2) = p(p-1)q$, we deduce that 3 also divides $\frac{p-1}{2} = \frac{n-1}{2}$. The possibility $3 = \frac{n-1}{2}$ is excluded by $n \geq 11$ and thus $\frac{n-1}{2} \notin P$.

Since $2P, P+1$ are subsets of the even positive integers, while $P, P+2, 2P+1$ are subsets of the odd positive integers, we have nine cases to deal with. For all those cases we need to understand the number and nature of the components of $\tilde{P}_0(A_n)$ and $P_0(\mathcal{T}(A_n))$ containing a vertex of order belonging to $B(n)$; the number and nature of components of $\mathcal{O}_0(A_n)$, different from Θ_n , containing a vertex belonging to $B(n)$. Once that is done, by Lemma 3.2, we immediately reach full information for the components of $P_0(A_n)$ too. The counting of $\tilde{c}_0(A_n)$ is carried on applying Procedure 3.5.

Let $n \in 2P \setminus (P+1)$. Then n is even, $\frac{n}{2} = p \in P$ and $n-1 \notin P$, so that the only critical order is p . But, by Lemma 6.5, p is a vertex of $\tilde{o}(\tilde{\Omega}_n)$ and thus $V_{\Theta_n} = O_0(A_n) = V_{\tilde{o}(\tilde{\Omega}_n)}$ and $c_0(\mathcal{O}(A_n)) = 1$. To examine the graph $\tilde{P}_0(A_n)$, let $[\pi] \in [A_n]_0$ with $o(\pi) = p$. Then $T_\pi \in \{[p^2], [1^p, p]\}$ and, by Lemma 6.5 the vertices of type $[1^p, p]$ are in $\tilde{\Omega}_n$, while those of type $T_1 = [p^2]$ are isolated. Thus $\tilde{c}_0(A_n)_{T_1} = \mu_{[p^2]}[A_n]$ and hence $\tilde{c}_0(A_n) = \frac{4(n-1)(n-3)!}{n} + 1$. Moreover $c_0(\mathcal{T}(A_n)) = 2$.

Let $n \in 2P \cap (P+1)$. Then n is even, $\frac{n}{2} = p \in P$ and $n-1 = q \in P$, so that the critical orders are the primes p and q . By Lemmas 6.5 and 4.1, we have that $p \in V_{\tilde{o}(\tilde{\Omega}_n)} \subseteq V_{\Theta_n}$ while q is isolated in $\mathcal{O}_0(A_n)$. Thus $c_0(\mathcal{O}(A_n)) = 2$ and, by Lemma 3.3, $V_{\Theta_n} = O_0(A_n) \setminus \{q\} = V_{\tilde{o}(\tilde{\Omega}_n)}$. To examine the graph $\tilde{P}_0(A_n)$, let first $[\pi] \in [A_n]_0$ with $o(\pi) = p$. As in the previous case, $T_\pi \in \{T_1 = [p^2], T = [1^p, p]\}$ and $T \in \mathcal{T}(\tilde{\Omega}_n)$ while $\tilde{c}_0(A_n)_{T_1} = \mu_{[p^2]}[A_n]$. Moreover each $[\pi] \in [A_n]_0$ with $o(\pi) = q$ is isolated and $T_\pi = [1, q]$. So, by Lemma 3.3, $T_2 = [1, q]$ is isolated in $P_0(\mathcal{T}(A_n))$ and $\tilde{c}_0(A_n)_{T_2} = n(n-3)!$. It follows that $\tilde{c}_0(A_n) = \frac{4(n-1)(n-3)!}{n} + n(n-3)! + 1$ and $c_0(\mathcal{T}(A_n)) = 3$.

Let $n \in (P+1) \setminus 2P$. Here n is even, $\frac{n}{2} \notin P$ and $n-1 = p \in P$, so that the only critical order is the prime p . By Lemma 4.1, p is isolated in $\mathcal{O}_0(A_n)$ and so $c_0(\mathcal{O}(A_n)) = 2$. Thus, by Lemma 3.3, we also get $V_{\Theta_n} = O_0(A_n) \setminus \{p\} = V_{\tilde{o}(\tilde{\Omega}_n)}$. Moreover, each $[\pi] \in [A_n]_0$ with $o(\pi) = p$ is isolated in $\tilde{P}_0(A_n)$. Thus $T_1 = [1, p]$ is isolated in $P_0(\mathcal{T}(A_n))$ and $\tilde{c}_0(A_n)_{T_1} = n(n-3)!$, so that $\tilde{c}_0(A_n) = n(n-3)! + 1$ and $c_0(\mathcal{T}(A_n)) = 2$.

Let $n \in P \setminus [(P+2) \cup (2P+1)]$. Here the only critical order is n , which by Lemma 4.1 is isolated, so that $c_0(\mathcal{O}(A_n)) = 2$. Then, by Lemma 3.3, $V_{\Theta_n} = O_0(A_n) \setminus \{n\} = V_{\tilde{o}(\tilde{\Omega}_n)}$. For $T_1 = [n]$, we get $\tilde{c}_0(A_n)_{T_1} = (n-2)!$, so that $\tilde{c}_0(A_n) = (n-2)! + 1$ and $c_0(\mathcal{T}(A_n)) = 2$.

Let $n \in (2P+1) \setminus [P \cup (P+2)]$. Here the only critical order is the prime $p = \frac{n-1}{2}$, which by Lemma 6.5, is a vertex of $\tilde{o}(\tilde{\Omega}_n)$. Thus $c_0(\mathcal{O}(A_n)) = 1$ and $V_{\Theta_n} = O_0(A_n) = V_{\tilde{o}(\tilde{\Omega}_n)}$. Moreover $T_1 = [1, p^2]$ is isolated in $P_0(\mathcal{T}(A_n))$ and $\tilde{c}_0(A_n)_{T_1} = \mu_{T_1}[A_n]$. Thus $\tilde{c}_0(A_n) = \frac{4n!}{(n-1)^2(n-3)} + 1$ and $c_0(\mathcal{T}(A_n)) = 2$.

Let $n \in (P+2) \setminus [P \cup (2P+1)]$. Here the only critical order is the prime $n-2$. By Lemmas 4.1 and 3.3, $n-2$ is isolated in $\mathcal{O}_0(A_n)$ while the type $[1^2, n-2]$ is isolated in $P_0(\mathcal{T}(A_n))$. Then $c_0(\mathcal{O}(A_n)) = 2$, $V_{\Theta_n} = V_{\tilde{o}(\tilde{\Omega}_n)}$, $\tilde{c}_0(A_n) = \frac{n(n-1)(n-4)!}{2} + 1$ and $c_0(\mathcal{T}(A_n)) = 2$.

Let $n \in P \cap (P+2)$. Here the critical orders are the primes $n, n-2$, both isolated in $\mathcal{O}_0(A_n)$, by Lemma 4.1. Moreover the types $[n], [1, n-1]$ are isolated in $P_0(\mathcal{T}(A_n))$. Thus $c_0(\mathcal{O}(A_n)) = 3$, $\tilde{c}_0(A_n) = (n-2)! + \frac{n(n-1)(n-4)!}{2} + 1$ and $c_0(\mathcal{T}(A_n)) = 3$. Applying Lemma 3.3 we also get $V_{\Theta_n} = V_{\tilde{o}(\tilde{\Omega}_n)}$.

Let $n \in P \cap (2P+1)$. The critical orders are the primes $n, \frac{n-1}{2}$. By Lemmas 4.1, 3.3 and 6.5, with the usual arguments, we get $c_0(\mathcal{O}(A_n)) = 2$, $V_{\Theta_n} = V_{\tilde{o}(\tilde{\Omega}_n)}$, $\tilde{c}_0(A_n) = (n-2)! + \frac{4n(n-2)(n-4)!}{n-1} + 1$ and $c_0(\mathcal{T}(A_n)) = 3$. Moreover the types $[n]$ and $[1, (\frac{n-1}{2})^2]$ are isolated in $P_0(\mathcal{T}(A_n))$.

Let $n \in (P+2) \cap (2P+1)$. The critical orders are the primes $n-2$ and $\frac{n-1}{2}$. By Lemmas 4.1 and 6.5, n is isolated in $\mathcal{O}_0(A_n)$ while $\frac{n-1}{2} \in V_{\tilde{o}(\tilde{\Omega}_n)}$. Thus $c_0(\mathcal{O}(A_n)) = 2$ and $V_{\Theta_n} = V_{\tilde{o}(\tilde{\Omega}_n)}$. Moreover the types $[1^2, n-2]$ and $[1, (\frac{n-1}{2})^2]$ are isolated in $P_0(\mathcal{T}(A_n))$, so that $\tilde{c}_0(A_n) = \frac{n(n-1)(n-4)!}{2} + \frac{4n(n-2)(n-4)!}{n-1} + 1$ and $c_0(\mathcal{T}(A_n)) = 3$. \square

Proof of Theorem B. The case by case analysis in the proof of Theorem A has shown many facts. In particular, $\tilde{\Omega}_n$ is the only possible component of $\tilde{P}_0(S_n)$ not reduced to an isolated vertex and $V_{\Theta_n} = V_{\tilde{o}(\tilde{\Omega}_n)}$. Since \tilde{o} is complete, by [3, Proposition 5.2], we then deduce $\Theta_n = \tilde{o}(\tilde{\Omega}_n)$.

Call now main component of $P_0(A_n)$, $\tilde{P}_0(A_n)$, $P_0(\mathcal{T}(A_n))$ and $\mathcal{O}_0(A_n)$, respectively, the component Ω_n such that $\pi(\Omega_n) = \tilde{\Omega}_n$ defined in Corollary 5.3, $\tilde{\Omega}_n$, $\tilde{t}(\tilde{\Omega}_n)$ and $\tilde{o}(\tilde{\Omega}_n)$. Note that $V_{\tilde{t}(\tilde{\Omega}_n)} = \mathcal{T}(\tilde{\Omega}_n)$

and that $\tilde{t}(\tilde{\Omega}_n)$ is a component because \tilde{t} is a pseudo-covering. By Corollary 5.3, no main component is complete.

We show now that, for every $X_0 \in \{\tilde{P}_0(A_n), P_0(\mathcal{T}(A_n))\}$, all the components except the main are isolated vertices of order a prime $p \in B(n)$. Let $\tilde{C} \in \tilde{\mathcal{C}}_0(A_n)$, with $\tilde{C} \neq \tilde{\Omega}_n$. Choose $[\psi] \in V_{\tilde{C}}$, so that $\tilde{C} = C([\psi])$. Since \tilde{t} is pseudo-covering, by Proposition 3.1 (ii) (b), we get $\tilde{t}(\tilde{C}) = \tilde{t}(C([\psi])) = C(T_\psi)$. But, from $[\psi] \notin V_{\tilde{\Omega}_n}$ we deduce, by Lemma 6.6, that $o(\psi) \in B(n)$ and thus, by our case by case analysis, T_ψ is isolated. By Lemma 3.3 (i), this is equivalent to $[\psi]$ isolated and so \tilde{C} is reduced to the vertex $[\psi]$. Next let $C \in \mathcal{C}_0(\mathcal{T}(A_n))$, with $C \neq \tilde{t}(\tilde{\Omega}_n)$. By Proposition 3.1 (ii) (b), we have $C = \tilde{t}(\tilde{C})$ for a suitable $\tilde{C} \in \tilde{\mathcal{C}}_0(A_n)$ and $\tilde{C} \neq \tilde{\Omega}_n$ and, by the above case, we know that \tilde{C} is an isolated vertex $[\psi]$ of $\tilde{P}_0(A_n)$. Thus, by Lemma 3.3 (i), $T_\psi \in V_C$ is an isolated vertex of $P_0(\mathcal{T}(A_n))$ and so C is reduced to the vertex T_ψ .

Now, to obtain that each component of $P_0(A_n)$ different from Ω_n is a complete graph on $p - 1$ vertices it is enough to invoke Lemma 3.2. Finally observe that the case by case analysis in the proof of Theorem A has shown that $|B(n)| \leq 2$, which says that the primes involved in components different from the main one are at most 2 for all the graphs in \mathcal{G}_0 . \square

Corollary 6.7. *The minimum $n \geq 4$ such that $P(A_n)$ is 2-connected is $n = 16$. There exists infinitely many n such that $P(A_n)$ is 2-connected.*

Proof. Let first $4 \leq n \leq 15$. Then we have $4 \leq n \leq 10$, or $n \geq 11$ and $n \in A$. Thus, by Theorem A, we get $c_0(A_n) > 1$. Moreover, since $16 \in \mathbb{N} \setminus A$, Theorem A implies that $c_0(A_{16}) = 1$. To prove that $\mathbb{N} \setminus A$ is infinite we show that it contains $B = \{4k^2 : k \in \mathbb{N}, k \geq 2\}$. Let $n = 4k^2 \in B$. Since n is even and $n \geq 16$, we have $n \notin P \cup (P+2) \cup (2P+1)$. Moreover $\frac{n}{2} = 2k^2$ is not a prime so that $n \notin 2P$. Finally, we have $n - 1 = (2k - 1)(2k + 1)$, so that $n \notin P + 1$. Hence $n \in \mathbb{N} \setminus A$. \square

Note that there exists also some n odd such that $P(A_n)$ is 2-connected. For instance this happens for $n = 51$.

Proof of Corollary C. It is immediate by checking that $c_0(\mathcal{T}(A_n)) = 1$ if and only if $c_0(A_n) = 1$, using Table 1 and Table 2, and noting that the natural numbers n such that $4 \leq n \leq 10$ are contained in A . To argue about the infinitely many $n \in \mathbb{N}$ such that $P(A_n)$ is 2-connected, we use Corollary 6.7. \square

We finally give a more concise overview on the number of components of the graph $\mathcal{O}_0(A_n)$.

Corollary 6.8. *The values of $c_0(\mathcal{O}(A_n))$ for $n \geq 4$, with $n \neq 6$, are given in Table 6 below.*

TABLE 6. $c_0(\mathcal{O}(A_n))$, for $n \geq 4$, $n \neq 6$.

| $c_0(\mathcal{O}(A_n))$ | $n \geq 4, n \neq 6$ |
|-------------------------|---|
| 1 | $n \notin P \cup (P+1) \cup (P+2)$ |
| 2 | $n \in [P \setminus (P+2)] \cup [(P+2) \setminus P] \cup (P+1)$ |
| 3 | $n \in P \cap (P+2)$ |

Proof. A check on the Tables 1 and 2. \square

Proof of Corollary D. It follows from Theorem 4.2 and Corollary 6.8. \square

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